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G Thompson

Department of Mathematics, The University of Toledo, 2801 W Bancroft Street, Toledo, OH 43606, USA

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Abstract. The two-dimensional symmetric connections whose geodesic equations are derivable from a Lagrangian function are divided into five classes. This classification is compared with that of Douglas for more general classes of systems of ordinary differential equations. Three of the five classes of connections are further investigated and in most cases specific Lagrangians are exhibited. In particular those connections that are engendered by Lagrangians homogeneous in velocities are characterized in terms of the Ricci tensor of each connection. Finally several examples of variational connections that possess integrals of motion are given, thereby extending the known class of completely integrable systems.

1. Introduction

The inverse problem of the calculus of variations is concerned with finding necessary and sufficient conditions for a given system of second-order ordinary differential equations to be derivable from a regular Lagrangian function. In [3] Douglas gave an exhaustive treatment of the two degrees of freedom case. In [1] various aspects of the inverse problem were considered. In particular, section 7 of [1] was concerned with finding Lagrangians for the geodesic system associated to a symmetric linear connection in two dimensions. This paper extends these investigations to the point where most connections for which at least one explicit Lagrangian exists are written down.

We have attempted to make this paper as self-contained as possible. In particular we do not discuss the Helmholtz conditions which provide necessary and sufficient conditions for the existence of Lagrangians. Besides [1, 3] the Helmholtz conditions are well studied in [2, 5], for example.

In section 2 we obtain a basic classification for variational connections. Again the discussion is brief and the reader will have to consult [1] for the full story. Of the five types of connection considered we ignore flat connections (type (V)) and another class (type (III)) with regard to the problem of exhibiting explicit Lagrangians. Even the existence of type (III) connections is at this point in doubt let alone the existence of Lagrangians for such connections.

The remaining three types of connections are those for which either the Ricci tensor K_{ij} is degenerate, or if non-degenerate there exists a Lagrangian which is a homogeneous function of the velocities u and v. Recall that in two dimensions the Ricci tensor embodies the entire curvature tensor. See [4] for more details. We use x and y for the position variables so that u and v rather than $\dot{x}$ and $\dot{y}$ are the derivatives of x and y, with respect to the independent variable t.

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Section 3 explains how the classification of section 2 is related to the well known classification of Douglas [3]. Sections 4–6 are each devoted to one of the classes of connections given in section 2. Explicit Lagrangians are exhibited in most cases.

Note that any Lagrangian function which is homogeneous of degree k in u and v, provided it is regular, gives Euler–Lagrange equations with right-hand sides that are homogeneous of degree 2 in u and v, but not necessarily homogeneous quadratic. As a result of sections 4 and 6 we obtain a characterization of those homogeneous Lagrangians that give rise to geodesic sprays. Furthermore the connections themselves are characterized by simple properties of the Ricci tensor.

2. The inverse problem for connections

We consider a symmetric connection in two dimensions whose geodesic equations we write as

$$\dot{u}^i + \Gamma^i_{jh} u^j u^h = 0 \tag{2.1}$$

where Γ^i_{jh} are functions of the coordinates (x^i) and u^i denotes the time derivative of x^i. The inverse problem for (2.1) is concerned with finding a function $L(t, x^i, u^i)$ so that (2.1) are the Euler–Lagrange equations of L.

In [1] the inverse problem for connections and much more general systems was considered. We shall present sufficient information of that theory so that the reader has a relatively complete understanding of the inverse problem as it relates to (2.1).

In [1] it was shown that the following condition is necessary for the existence of a Lagrangian that engenders (2.1) where K_{ij} denotes the Ricci tensor of the Γ^i_{jh}:

$$K_{(k\hat{i}} K_{h\hat{j};l)} = K_{(k\hat{j}} K_{h\hat{i};l)}. \tag{2.2}$$

A very similar condition, namely,

$$K_{(k\hat{i}} K_{hj);l} = K_{(hj} K_{k)i;l} \tag{2.3}$$

was also shown to be of significance. In fact in full (2.3) comprises six conditions whereas (2.2) comprises four conditions and (2.3) implies (2.2).

We can now give a classification of variational connections into types that we number (I)–(V).

(I) K_{ij} degenerate and (2.3) satisfied;
(II) K_{ij} non-degenerate and (2.3) satisfied;
(III) K_{ij} non-degenerate and (2.2) but not (2.3) satisfied;
(IV) K_{ij} skew-symmetric but not zero;
(V) K_{ij} zero.

From the theory developed in [1] it is known that every variational connection belongs to one of the five types. Clearly for type (V) the associated connection is flat and hence variational as too are the connections of types (I)–(IV). For type (III) it is not known whether such connections actually exist. (For more details we refer to [1 p 95] and the numbering (I), (II) and (III) is chosen to agree with that reference. The existence problem for type (III) connections leads to a delicate problem in differential systems theory which we hope to revisit in the future. However, even if this problem can be answered affirmatively, it seems doubtful whether concrete examples of this type can be written down.) The remainder of this paper will not be concerned with the connections of types (III) or (V).

3. Relation to the classification of Douglas

As an alternative to the theory of [1] we summarize the situation for connections in terms of Douglas' classification [3]. It must be emphasized that Douglas also used capital Roman numerals (I)–(IV) in his classification but that it is different to the one used here in section 2 which refers to connections only. First of all a connection belongs to case (I) of Douglas if and only if it is flat. Secondly, case (IV) of Douglas can never happen (the fundamental 3×3 matrix is always singular) and in Douglas' case (III) only (IIIb) can occur in which case a Lagrangian is necessarily singular. Finally equation (2.2) above is precisely the condition for a connection to belong to case (II) of Douglas.

As regards type (II) of section 2 above the system (2.1) falls precisely into cases (II) a1, (II) a2 and (II)b of Douglas according as the Ricci tensor is degenerate, non-degenerate but not skew-symmetric, or skew-symmetric, respectively. In the remainder of this paper we shall have no need to refer to Douglas' classification and the Roman numerals used below will always pertain to section 2 above.

4. Connections with Ricci skew-symmetric

In [1] it was shown that for type (IV) connections coordinates could be introduced relative to which the geodesic equations are

$$\dot{u} = -\varphi_x u^2 \qquad \dot{v} = \varphi_y v^2 \tag{4.1}$$

for some function φ of x and y and for which φ_{xy} is non-zero, at least on an open subset of the ambient manifold. Moreover equations (4.1) are engendered by the Lagrangian

$$L = \mathrm{e}^{\varphi}\frac{u}{v}. \tag{4.2}$$

Indeed the second Helmholtz condition, which expresses the self-adjointness of the Jacobi endomorphism to the multiplier matrix, see [1, 2, 5], reduces since K_{ij} is skew to

$$u^2 L_{uu} + 2uvL_{uv} + v^2 L_{vv} = 0. \tag{4.3}$$

From (4.3) it follows that L may be written in the form

$$L = u + \lambda v \tag{4.4}$$

where μ and λ are functions of x, y and z, respectively, and z is defined to be u/v. If one now computes the Euler–Lagrange equations of (4.4) one finds that they are given by (4.1) if and only if

$$\mu_x - z\mu_z\varphi_x = 0 \tag{4.5}$$

$$\mu_y - z\mu_z\varphi_y = 0 \tag{4.6}$$

$$(z^2\varphi_x + z\varphi_y)\lambda_{zz} + \lambda_x - z\lambda_{zx} - \lambda_{zy} = 0. \tag{4.7}$$

From (4.5) and (4.6) one may deduce that

$$\varphi_x\mu_y - \varphi_y\mu_x = 0. \tag{4.8}$$

But now (4.5), (4.6) and (4.8) simply imply that μ is a smooth function of the Lagrangian given by (4.2). Any such Lagrangian is necessarily regular unless it is constant. The most general Lagrangian for (4.1) is thus obtained by adding to a function of L given by (4.2) any solution of (4.7).

At this point and for the only time in this paper we consider some questions of a global nature. Note that (4.2) may be defined on the complement of the zero section in the tangent bundle TM of the two-dimensional manifold that supports the connection given by (4.1). Alternatively L may be considered as being defined on PTM, the bundle obtained by projectivizing the fibres of TM. However, it turns out that the induced connection on PTM is flat.

The Lagrangian (4.2) also shows how to define a connection with skew Ricci on the torus. If we take φ to be periodic in x and y, for example,

$$\varphi = \sin x \sin y \tag{4.9}$$

then the corresponding connection passes to the torus. In this case the Ricci tensor, although it is skew-symmetric, does have some singularities.

5. Connections of type (I)

Turning next to connections of type (I) we shall obtain a different characterization of them for which we shall have to make a new definition. For connections that are not of type (IV) and (V) we define a one-form λ by

$$N\lambda = K_{ij}u^i\sigma^j \tag{5.1}$$

where

$$\sigma^j = \mathrm{d}u^j + \Gamma^j_{ik}u^i\,\mathrm{d}x^k \tag{5.2}$$

and

$$N = K_{ij}u^iu^j. \tag{5.3}$$

It may be shown that

$$N^2\,\mathrm{d}\lambda = (K_{rs}K_{ti;j} - K_{ri}K_{st;j})u^ru^su^t\,\mathrm{d}x^j \wedge \sigma^i. \tag{5.4}$$

Hence (2.3) is the condition for λ to be closed.

Theorem 5.1. A non-flat connection ∇ is of type (I) if and only if it has a parallel one-form.

Proof. If ∇ admits a parallel one-form, its geodesic equations may be written

$$\dot{u} = -(au^2 + 2buv + cv^2) \qquad \dot{v} = 0 \tag{5.5}$$

the one-form being $\mathrm{d}y$. An easy calculation shows that the first column of Ricci is zero. Furthermore λ may be shown to be $\frac{\mathrm{d}v}{v}$ hence λ is closed and (2.3) is satisfied.

Conversely, if K is degenerate we may introduce coordinates so that either K_{11} and K_{21} are zero or K_{11} and K_{12} are zero. In fact the second case cannot happen as follows from proposition 7.8 in [1]. In the first case if we let the geodesics of the connection be given by

$$\dot{u} = -(au^2 + 2buv + cv^2) \qquad \dot{v} = -(\gamma u^2 + 2\beta uv + \alpha v^2) \tag{5.6}$$

one finds

$$\lambda = \mathrm{d}(\ln v) + z(\lambda\,\mathrm{d}x + \beta\,\mathrm{d}y) + \beta\,\mathrm{d}x + \alpha\,\mathrm{d}y \tag{5.7}$$

where z is again $\frac{u}{v}$.

From (5.7) we conclude that

$$\mathrm{d}\lambda = \mathrm{d}z \wedge (\gamma\,\mathrm{d}x + \beta\,\mathrm{d}y) + z(\beta_x - \gamma_y)\,\mathrm{d}x \wedge \mathrm{d}y + (\alpha_x - \beta_y)\,\mathrm{d}x \wedge \mathrm{d}y \tag{5.8}$$

and hence λ is closed if and only if β and γ both vanish and α is independent of x. However, then a transformation of the form

$$\bar{x} = \bar{x}(x, y) \qquad \bar{y} = \bar{y}(y) \tag{5.9}$$

can be used to eliminate α and reduce (5.6) to (5.5). □

Note that if the connection is of the form given by (5.5) a further coordinate transformation can be made so as to eliminate a, b or c. Let us assume then that c is also zero in (5.5). In this case the self-adjointness Helmholtz condition becomes

$$uL_{uu} + vL_{uv} = 0. \tag{5.10}$$

From (5.10) we conclude that there exist smooth functions θ and ψ such that

$$L = v\theta(x, y, z) + \psi(x, y, v). \tag{5.11}$$

Now using the fact that $uL_u + vL_v - L$ is a first integral of the geodesics, by modifying θ if necessary, we may assume that ψ depends on v only. In order for L to be a regular Lagrangian it is necessary for ψ to be a nonlinear function of v. In case it is nonlinear, a straightforward calculation shows that (5.11) gives rise to the geodesic equations (5.5) if and only if θ satisfies

$$(zx^2 + 2bz + c)\theta_{zz} + \theta_x - z\theta_{zx} - \theta_{yz} = 0. \tag{5.12}$$

The most general Lagrangian L is then obtained by solving (5.12) and choosing ψ to be a nonlinear function of v.

A special case of type (I) connections is worth noting, namely, those for which the Ricci tensor is degenerate and symmetric. Indeed with reference to (5.5) one finds that Ricci is symmetric if and only if

$$a_y - b_x = 0. \tag{5.13}$$

From (5.13) it follows that there exists a function $\varphi(x, y)$ such that

$$a = \varphi_x \qquad b = \varphi_y. \tag{5.14}$$

We summarize the situation by means of the following theorem.

Theorem 5.2. A connection of type (I) for which the Ricci tensor is symmetric has geodesic equations given by

$$\dot{u} = -\varphi_x u^2 - 2\varphi_y uv \qquad \dot{v} = 0. \tag{5.15}$$

Furthermore these geodesics are derived from the Lagrangian

$$L = \mathrm{e}^{2\varphi(x,y)}\frac{u^2}{v} + \psi(v) \tag{5.16}$$

where ψ_{vv} is non-zero.

Proof. As we have stated, we may transform away the coefficient c in (5.5) by a coordinate transformation. Equation (5.15) now follows from (5.14). Finally note in this case $\theta = \mathrm{e}^{2\varphi}z^2$ gives a solution to (5.12). □

6. Connections of type (II) and homogeneous Lagrangians

It remains to consider type (II) Lagrangians. We introduce the differential operator Δ by

$$\Delta = u\frac{\partial}{\partial v} + v\frac{\partial}{\partial v}. \tag{6.1}$$

A Lagrangian function will then be homogeneous of degree k (in velocities) provided

$$\Delta L = kL. \tag{6.2}$$

Note that in order for L to be regular it is necessary that $k \neq 1$ and that if L is independent of time t it is a first integral of its own Euler–Lagrange equations.

Suppose now that the Euler–Lagrange equations of L are the geodesics of a symmetric linear connection Δ. The self-adjointness Helmholtz condition gives

$$(K_{12}u + K_{22}v)\Delta(L_u) = (K_{11}u + K_{12}v)\Delta(L_v). \tag{6.3}$$

From the definition of the one-form λ given by (5.1) we find that

$$\lambda = \frac{\left(\Delta\left(\frac{\partial L}{\partial u^i}\right)\right)\sigma^i}{\Delta(\Delta - 1)L}. \tag{6.4}$$

In the special case where L satisfies (6.2) with k different from zero or unity it follows that

$$\lambda = \frac{\mathrm{d}(\ln L)}{k}. \tag{6.5}$$

It was shown in [1, proposition 7.8] that whenever λ is closed the local function L of which it is the differential is a Lagrangian for the given geodesic system. Furthermore this Lagrangian is regular provided that Ricci is regular (and not skew-symmetric). In fact the Hessian of L is given up to a factor of KN^{-4} by

$$(K_{11})^2u^4 + 2K_{11}K_{12}u^3v + ((K_{12} + K_{21})^2 + 2K_{11}K_{22})u^2v^2 + 2K_{22}K_{21}uv^3 + (K_{22})^2v^4$$

where K is the determinant of K_{ij}.

Our next objective is to show that a connection is of type (II) only if its geodesic equations are derivable from a Lagrangian which is homogeneous of degree k where k is not zero. We assume that our Lagrangian L may be written as

$$L = v^p M(x, y, z) \tag{6.6}$$

where z stands for u/v and M is a smooth function of its three arguments. Here p is a real number so (6.6) expresses the fact that L is homogeneous of degree p. We next compute the Euler–Lagrange equations for the Lagrangian given by (6.6) and demand that they are the geodesic equations of a spray so that the right-hand sides are quadratic in u and v. By factoring out v^2 we have two expressions Q and R which are quadratic in z. One finds that M must satisfy the following equations in order that (6.6) should give rise to a spray:

$$\frac{(zM_{zx} + M_{yz} - M_x)M_z - (zM_x + M_y)M_{zz}}{pMM_{zz} - (p-1)M_z^2} = R \tag{6.7}$$

$$\frac{pM(M_x - M_{zy} - zM_{zx}) + (p-1)M_z(zM_x + M_y)}{pMM_{zz} - (p-1)M_z^2} = Q - zR. \tag{6.8}$$

These last two equations may be rearranged so as to give

$$zM_x + M_y = R(zM_z - pM) - QM_z \tag{6.9}$$

$$zM_{zx} + M_{yz} - M_x = R(zM_{zz} - (p-1)M_z) - QM_{zz}. \tag{6.10}$$

Differentiating (6.9) with respect to z gives

$$zM_{zx} + M_{yz} + M_x = R_z(zM_z - pM) + R(zM_{zz} - (p-1)M_z) - QM_{zz} - Q_zM_z. \tag{6.11}$$

From (6.9)–(6.11) we obtain

$$M_x = \tfrac{1}{2}R_z(zM_z - pM) - \tfrac{1}{2}Q_zM_z \tag{6.12}$$

$$M_y = \tfrac{1}{2}(zM_z - pM)(2R - zR_z) + \tfrac{1}{2}M_z(zQ_z - 2Q). \tag{6.13}$$

From (6.12) we compute M_{xy} and M_{yx} from (6.13) and equate these two. After some elimination and using (6.9) and (6.11) we obtain the following condition:

$$\begin{aligned}
2(zR_{zx} + R_{zy} - 2R_x)(zM_z - pM) - 2(Q_{zy} + zQ_{zx} - 2Q_x)M_z&\\
+((zR_z - Q_z)R_z + 2(Q - zR)R_{zz})(zM_z - pM)&\\
-(2RQ_z - 2R_zQ + 2(Q - zR)Q_{zz} - Q_z(Q_z - zR_z))M_z = 0.&
\end{aligned} \tag{6.14}$$

It turns out that equation (6.14) may be expressed in terms of the components K_{ij} of the Ricci tensor corresponding to the connection components contained in Q and R. Indeed one finds that

$$(K_{11}z^2 + (K_{12} + K_{21})z + K_{22})M_z = p(K_{11}z + K_{21})M. \tag{6.15}$$

Equation (6.15) is an ordinary differential equation for M in the variable z. There are essentially three cases to consider depending on whether the coefficient of M_z in the left-hand side of (6.15) (i) factors into distinct linear factors, (ii) repeated linear factors or (iii) is an irreducible quadratic. Geometrically, these cases are distinguished by the signature of the symmetric part of K_{ij}. In case (i) $K_{(ij)}$ is non-singular and indefinite, in case (ii) $K_{(ij)}$ is singular and finally in case (iii) $K_{(ij)}$ is definite. Before investigating these cases further we prove two lemmas.

Lemma 6.1. The Lagrangian L given by (6.6) is degenerate if and only if L is a power of a function which is homogeneously linear in u and v. In particular a Lagrangian homogeneous of degree 0 is always regular, provided it depends on z.

Proof. As we see in (6.7) and (6.8), L is degenerate if and only if M satisfies

$$pMM_{zz} - (p-1)M_z^2 = 0. \tag{6.16}$$

Note that if p is zero (6.16) is satisfied if and only if M_z is zero. Equation (6.16) is easily integrated giving

$$Mv^p = (a(x,y)u + b((x,y)v)^p$$

for some smooth functions a and b of x and y. The converse is well known. □

Lemma 6.2. The function $(au + bv)^p(cu + dv)^q$ where a, b, c, d, p, q are functions of x, y such that $ad - bc$ is non-zero is locally equivalent to $F(\bar{x}, \bar{y})\bar{u}^p\bar{v}^q$ where $(x, y) \to (\bar{x}, \bar{y})$ is a non-singular change of variables and $(\bar{u}, \bar{v})$ changes from (u, v) by the Jacobian of $(x, y) \to (\bar{x}, \bar{y})$.

Proof. Consider the following algebro-differential system for (x, y, F):

$$\begin{aligned}
&b\bar{x}_x - a\bar{x}_y = 0\\
&d\bar{y}_x - c\bar{y}_y = 0\\
&a^pc^q = F\bar{x}_x^p\bar{y}_x^q.
\end{aligned} \tag{6.17}$$

The first two equations in (6.17) may be solved for x and y and then the third used to determine F. Provided that $(x, y) \to (\bar{x}, \bar{y})$ is non-singular, this gives the required transformation.

If x and y are functionally dependent we have

$$bc(\bar{x}_x\bar{y}_y - \bar{x}_y\bar{y}_x) = (ad - bc)\bar{x}_x\bar{y}_y = 0. \tag{6.18}$$

Since $ad - bc$ is non-zero we may assume without loss of generality that $\bar{x}_y$ is zero. Returning to the first equation in (6.17) we deduce that either x is constant or b is zero. However, we can always choose x so as not to be constant and if b is zero we may set $\bar{x} = x$. Since x and y are functionally dependent we must have $\bar{y}_y = 0$ and hence from the second equation in (6.17) we must have $d = 0$. However, if b and d are both zero there is nothing to prove.

Let us now reconsider case (i) of (6.15). An elementary integration shows that Mv^p is of the form $(au + bv)^p(cu + dv)^q$ where a, b, c, d, q are functions of x and y and $ad - bc$ is non-zero. Hence lemma 6.2 may be applied. A similar analysis in each of cases (ii) and (iii) may be made using a variation of lemma 6.2.

There is one further refinement concerning the form of the Lagrangian that can be made. In case of (i) of (6.15) we now know that the Lagrangian may be written in the form $\mathrm{e}^b u^r v^s$ where $r + s = p$. We shall show that in fact both r and s have to be constant. Indeed the Euler–Lagrange equations for $\mathrm{e}^b u^r v^s$ turn out to be:

$$\begin{aligned} r(1 - r - s)\dot{u} &= [(r + s - 1)(b_x + r_x \ln u + s_x \ln v) + rs_x]u^2 - (s - 1)r_y uv \\ s(1 - r - s)\dot{v} &= [(r + s - 1)(b_y + r_y \ln u + s_y \ln v) + sr_y]v^2 - (r - 1)s_x uv \end{aligned} \tag{6.19}$$

and since $r + s$ is constant it easily follows that we can only have r_x, r_y, s_x and s_y all zero in order for (6.19) to represent the geodesic equations of a spray. Hence r and s are both constants. Similar arguments may be made in the case (ii) and (iii) of (61.5) and we summarize the various possibilities as follows. □

Theorem 6.3. A geodesic spray with non-zero curvature arises from an homogeneous Lagrangian if and only if the associated connection is of type (II) or type (IV). For type (II) connections there are three cases respectively as the symmetric part of Ricci is non-degenerate and indefinite, degenerate (but not zero), or (positive or negative) definite; in each of these respective cases there is a canonical Lagrangian and hence canonical form of the geodesic equations of the following type:

$$\left.\begin{array}{lll} L = \mathrm{e}^b u^r v^s & \dot{u} = \dfrac{-b_x}{r} u^2 & \\ (b = b(x, y)) & r, s \in (\mathbb{R}) & \dot{v} = \dfrac{-b_y}{s} v^2 \end{array}\right\} \tag{6.20}$$

$$\left.\begin{array}{ll} L = \mathrm{e}^v v^2 \mathrm{e}^{\frac{u}{v}} & \dot{u} = -2b_x uv - (b_y - 2b_x)v^2 \\ (b = b(x, y)) & \dot{v} = -b_x v^2 \end{array}\right\} \tag{6.21}$$

$$\left.\begin{array}{ll} L = (u^2 + v^2)\mathrm{e}^{b + c\arctan(\frac{u}{v})} & \dot{u} = \dfrac{1}{4 + c^2}[(b_y c - 2b_x)(u^2 - v^2) - 2(b_x c + 2b_y)uv] \\ (c \in \mathbb{R}, b = b(x, y)) & \dot{v} = \dfrac{1}{4 + c^2}[(b_x c + 2b_y)(u^2 - v^2) + 2(b_y c - 2b_x)uv]. \end{array}\right\} \tag{6.22}$$

7. Completely integrable connections

In this section we give several examples of variational connections that possess explicit integrals of motion and therefore constitute new non-standard examples of completely integrable systems when they are transformed into the Hamiltonian picture.

In particular, it should be noted that Killing's equations for homogeneous integrals of motion makes sense for the geodesics of a connection irrespective of whether the connection is engendered by a pseudo-Riemannian metric.

Example 7.1. In the Lagrangian (5.16) we set

$$\psi(v) = \tfrac{1}{2}v^2. \tag{7.1}$$

There is a qualitative difference between those systems for which $\varphi_{yy} + \varphi_y^2$ is or is not independent of x. Invariantly this difference corresponds to the tensor $B_{ijk}(= K_{ij;k} - K_{kj;i})$ being zero or not. In the latter case v is the unique linear integral up to scaling by a constant.

In the former case there are linearly independent integrals. To exhibit these integrals we rewrite the geodesics in the following form where A is a function of y and B and C are functions of x:

$$\dot{u} = -\frac{AB'' + C''}{AB' + C'}u^2 - \left(\frac{2A'B'}{AB' + C} - \frac{A''}{A'}\right)uv \qquad \dot{v} = 0. \tag{7.2}$$

The integrals of motion are v, $\frac{(AB'+C')}{A'}u + Bv$ and $\frac{A(AB'+C')}{A'}u - Cv$.

Example 7.2. Consider the Lagrangian given by (i) of theorem 6.3. There is a qualitative difference between the cases where r and s are or are not equal. In the former case Ricci is symmetric so we are essentially in the Lorentz metric Lagrangian situation. In the latter case Ricci is not symmetric and from the Killing theory we conclude that there are at most two linearly independent degree y integrals of motion. A more detailed study reveals that there is at most one, although the argument is rather involved.

A fairly general class of such systems having one integral can be described as follows. Suppose that b satisfies

$$r\mathrm{e}^{\frac{b}{s}}b_x + s\mathrm{e}^{\frac{b}{r}}b_y = 0. \tag{7.3}$$

Then $\mathrm{e}^{\frac{b}{r}}u + \mathrm{e}^{\frac{b}{3}}v$ is an integral. Equation (7.3) may be solved by defining

$$\ln w = \frac{b}{r} - \frac{b}{s} \tag{7.4}$$

from which (7.3) implies that

$$rw_x + sww_y = 0. \tag{7.5}$$

The solution of (7.5) is given implicitly by

$$sxw - ry = F(w) \tag{7.6}$$

where F is an arbitrary function of w.

Example 7.3. Consider the Lagrangian given by (ii) of theorem 6.3 but suppose now that b is independent of y. Then one may check that $\frac{2}{v} - \frac{u}{v^z}$ is a first integral of the geodesics.

More generally examples of integrable systems may be obtained from Neother's theorem by assuming some invariance of the Lagrangian. For example for the Lagrangian $\mathrm{e}^b u^r v^s$ we could choose b to be a function of $px + qy$ where p and q are real. In this case $\frac{q}{u} - \frac{p}{v}$ is a first integral.

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